SEQUENCE AND SERIES

Sequence: A set of numbers whose domain is real number is called a **SEQUENCE** and the sum of a sequence is called a **SERIES.** If $a_1, a_2, a_3, a_4, \ldots\ldots, a_n, \ldots\ldots$ is a sequence, then the expression $a_1 + a_2 + a_3 + a_4 + a_5 + \ldots. + a_n + \ldots$ is a series. A series is finite or infinite according as the number of terms in the corresponding sequence is finite or infinite.

Progressions: It is not necessary that the terms of a sequence always follow a certain pattern or they are described by some explicit formula for the nth term. Those sequences whose terms follow certain patterns are called progressions.

For example 1, 4, 7, 10, 13 …….
7, 4, 1, – 2, – 5………
1, 2, 4, 8, 16………
8, 4, 2, 1, ½…. …… are each a sequence.

Also $f(n) = n^2$ is a sequence, then $f(1) = (1)^2 = 1$, $f(2) = 2^2$ 4, $f((3)) = (3)^2 = 9$ $f(10) = 10^2 = 100$ and so on.

The nth term of a sequence is usually denoted by T_n

Thus T_1 = first term, T_2 = second term, T_{10} tenth term and so on.

ARITHMETIC PROGRESSION (A.P.)

It is series of numbers in which every term after first can be derived from the term immediately preceding it by adding to it a fixed quantity called **Common Difference**. In general, the difference of successive terms is same.

Therefore $a_{n+1} - a_n$ = constant (= d) for all $n \in N$

Example:

- 1, 4, 7, 10, ……. Is an A. P. whose first term is 1 and the common difference is 4 – 1 = 3.
- 11, 7, 3, –1 …… is an A. P. whose first term is 11 and the common difference 7 – 11 = – 4.

If in an A. P. **a** = the first term,
d = common difference
$\mathbf{T_n}$ = the nth term
l = the last term,
$\mathbf{S_n}$ = Sum of the n terms.

Then a, a + d, a + 2d, a + 3d,….. are in A.P.

General term of an A. P $T_n = a + (n - 1) d$

Sum of first n terms of A.P. $S_n = n/2 (a + l)$ **or** $S_n = n/2 [2a + (n - 1) d]$

PROPERTIES OF A.P.

Prop I: If each terms of an AP is increased, decreased, multiplied or divided by the same non-zero number, then the resulting sequence is also an AP.

Prop II: In an AP, the sum of terms equidistant from the beginning and end is always same and equal to the sum of first and last terms.

Prop III: Three numbers in AP are taken as a – d, a, a + d.
Four numbers in AP are taken as a – 3d, a – d, a + d, a + 3d.
Five numbers in AP are taken as a – 2d, a – d, a, a + d, a + 2d.

Prop IV: Any term of an AP is equal to half the sum of the terms which are equidistant from it, i.e.

$$A_n, = \frac{1}{2} (a_{n-k}, + a_{n+k}), \quad \text{Where } k < n.$$

Prop V: The simple arithmetic mean (AM) of two numbers a and b is $\frac{a+b}{2}$.

Prop. VI: Three numbers a, b, c are in A.P. iff **2b = a + c.**

Proof: First, let a, b, c be in A.P. then b – a = common difference and c – b = common difference
$\Rightarrow$ b – a = c– b $\Rightarrow$ 2b = a + c

Conversely, let a, b, c be three numbers such that 2b = a + c.
hen we have to show that a, b, c are in A.P.
We have, 2b = a + c $\Rightarrow$ b – a = c – b $\Rightarrow$ a, b, c are in A.P.

Example: If 2/3, k, 5/8 are in A.P., find the value of k.

Solution: 2/3, k, 5/8 are in A.P. $\Rightarrow$ 2k = 2/3 + 5/8 $\Rightarrow$ 2k = 31/24 $\Rightarrow$ k = 31/48.

Prop. VII: A sequence is an A.P. iff its nth term is a linear expression in n i.e. a_n = An + B, where A, B are constants. In such a case the coefficient of n in a_n is the common difference of the A.P.

Prop. VIII: A sequence is an A.P. iff the sum of its first n terms is of the form An^2 + Bn, where A, B are constants independent of n. In such a case the common difference is 2A i.e. 2 times the coefficient of n^2.

Prop. IX: If the terms of an A.P. are chosen at regular intervals, then they form an A.P.

Prof. X: If a_n, a_{n+1} and a_{n+2} are three consecutive terms of an A.P., then $2a_{n+1} = a_n + a_{n+2}$.

<u>n^{th} term of an A.P. from the end :</u>

Let a be the first term and d be the common difference of an A.P. having m terms.
Then n^{th} term from the end is (m – n + 1)th term from the beginning.
So, nth term from the end = am – n + 1 = a + (m – n + 1) d

$$= T_{m-n+1} = a + (m - n)d$$

Ex : In the arithmetic progressions 2, 5, 8, …. upto 50 terms and, 3, 5, 7, 9, ….. upto 60 terms, find how many terms are identical ?

Sol. Let the m^{th} term of the first A.P. be equal to the n^{th} term o the second A. P. Then,
2 + (m – 1) × 3 = 3 + (n – 1) × 2 $\Rightarrow$ 3m – 1 = 2n +1

$$\Rightarrow 3m = 2n + 2 \Rightarrow \frac{m}{2} = \frac{n+1}{3} = k(\text{say})$$

$\Rightarrow$ m = 2k and n = 3k –1 $\Rightarrow$ 2k $\leq$ 50 and 3k – 1 $\leq$ 60 [$\therefore$ m $\leq$ 50 and n $\leq$ 60]

$$\Rightarrow k \leq 25 \quad \text{and} \quad k \leq 20\frac{1}{3} \Rightarrow k \leq 20 \qquad [\because k \text{ is a natural number}]$$

$\Rightarrow$ k = 1, 2, 3, ……… 20

Ex. Find the number of terms common to the two A.P.'s 3, 7, 11, …. 407 and 2, 9, 16, …. 709.

Sol. Let the number of terms in two A.P.'s be m and respectively. Then 407 = m^{th} term of first A.P., and 709 = n^{th} term of second
$\Rightarrow$ 407 = 3 + (m – 1) × 4 and 709 = 2 + (n – 1) × 7
$\Rightarrow$m = 102 and n = 102 So, each A.P. consists of 102 terms.
Let p^{th} term of first A.P. identical to q^{th} term of the second A.P. Then,
3 + (p – 1) × 4 = 2 + (q – 1) × 7 $\Rightarrow$ 4p – 1 = 7q – 5

$\Rightarrow$ 4p + 4 = 7q

$\Rightarrow 4(p+1) = 7q \Rightarrow \frac{p+1}{7} = \frac{q}{4} = k$ (say)

$\Rightarrow$ p = 7k – 1 and q = 4k

Since each A.P. consists of 102 terms, therefore

$p \le 102$ and $q \le 102$

$\Rightarrow 7k - 1 \le 102$ and $4k \le 102 \Rightarrow k \le 14\frac{5}{7}$ and $k \le 25\frac{1}{2}$

$\Rightarrow k \le 14 \Rightarrow k = 1, 2, 3, \ldots\ldots, 14$

Corresponding to each value of k, we get a pair of identical terms. Hence, there are 14 identical terms in two A.P.'s.

Note :

- If $a_1, a_2, a_3, \ldots.. a_n$ are in A.P., where $a_i > 0$ for all i. then

1. $$\frac{1}{\sqrt{a_1}+\sqrt{a_2}} + \frac{1}{\sqrt{a_2}+\sqrt{a_3}} + \ldots\ldots\ldots + \frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}} = \frac{n-1}{\sqrt{a_1}+\sqrt{a_n}}$$

2. $$\frac{1}{a_1a_n} + \frac{1}{a_2a_{n-1}} + \frac{1}{a_3a_{n-2}} + \ldots\ldots + \frac{1}{a_na_1} = \frac{2}{a_1+a_n}\left(\frac{1}{a_1} + \frac{1}{a_2} + \ldots. + \frac{1}{a_n}\right)$$

ARITHMETIC MEAN

Insertion of n arithmetic means between two given quantities. If between two given quantities a and b we have to insert n quantities $A_1, A_2, \ldots\ldots A_n$ such that a, $A_1, A_2, \ldots\ldots A_n$, b form an AP, then $A_1, A_2, \ldots\ldots A_n$ are called n Arithmetic means between a and b. This sequence consists of (n + 2) terms with first term a and last term b.

$\therefore$ b = a + (n + 2 – 1) d $\Rightarrow$ $d = \frac{b-a}{n+1}$

$\therefore A_1 = a + d, A_2 = a + 2d, \ldots\ldots. A_n = a + nd.$

On substituting the values of d we can find all Arithmetic means.

Also, sum of n A.M's between a and b = $\frac{n(a+b)}{2}$.

Some General formulae (to be crammed thoroughly)

1. Sum of first n natural numbers = $\sum n = \frac{n(n+1)}{2}$
2. Sum of first n odd natural numbers = $\mathbf{n^2}$
3. Sum of first n even natural numbers = **n (n + 1**)
4. Sum of square of first natural numbers = $\sum n^2 = \frac{n(n+1)(2n+1)}{6}$
5. Sum of cubes of first n natural numbers = $\sum n^3 = (\sum n)^2 = \left(\frac{n(n+1)}{2}\right)^2$

GEOMETRIC PROGRESSION (G.P.)

A series in which each preceding term is formed by multiplying it by a constant factor is called a Geometric Progression of G. P. The constant factor is called the common ratio and is formed by dividing any term by the term which precedes it.

In other words, a sequence, $a_1, a_2, a_3, \ldots, a_n, \ldots$ is called a geometric progression if $\frac{a_{n+1}}{a_n}$ = constant for all $n \in N$

The General form of a G. P. with n terms is $a, ar, ar^2, \ldots\ldots ar^{n-1}$

Thus if **a** = the first term

r = the common ratio,

T_n = then nth term and

S_n = the sum of n terms,

General term $T_n = ar^{n-1}$

Sum of first n terms of G.P. $S_n = \frac{a(r^n - 1)}{r - 1}$ **where r > 1**

$S_n = \frac{a(1 - r^n)}{1 - r}$ **where r < 1**

$S_n = an$ **where r = 1**

Sum to infinite G.P:

If a G.P. has **infinite terms** and **–1 < r < 1**, then sum of infinite G.P is $S_\infty = \frac{a}{1 - r}$

PROPERTIES OF G.P.

Prop I: If each term of a GP is multiplied or divided by the same non-zero quantity, then the resulting sequence is also a GP.

Prop II: **SELECTION OF TERMS IN G.P.**

Sometimes it is required to select a finite number of terms in G.P. It is always convenient if we select the terms in the following manner :

No. of terms	Terms	Common ratio
3	$\frac{a}{r}, a, ar$	r
4	$\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$	r^2
5	$\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$	r

If the product of the numbers is not given, then the numbers are taken as $a, ar, ar^2, ar^3, \ldots$

Prop III: Three non-zero numbers a, b, c are in G.P. iff $b^2 = ac$

Prop IV: If $a_1, a_2, a_3, \ldots a_n, \ldots$ is a G.P. of non-zero non-negative terms, then $\log a_1, \log a_2, \ldots, \log a_n, \ldots$ is an A.P. and vice-versa.

Prop V: The geometric mean between two members a and b is G = $\sqrt{(ab)}$

If $a_1, a_2, a_3, \ldots$ an are n non-zero, non-negative numbers, then their geometric mean G is given by

$G = (a_1, a_2, a_3, \ldots a_n)^{1/n}$

Ex. If p, q, r are in A.P. show that the pth, qth and rth terms of any G.P. are in G.P.

Sol. Let A be the first term and R the common ratio of a G.P. Then, $a_p = AR^{p-1}$, $a_q = AR^{q-1}$ and $a_r = AR^{r-1}$

We want to prove that a_p, a_q, a_e are in G.P. For this it is sufficient to show that $(a_q)^2 = a_p.\ a_e$.

Now $(a_q)^2 = (AR^{q-1})^2 = A^2R^{2q-2} = A^2R^{p+r-2}$ [$\therefore$ p, q, r are in A.P. $\therefore$ 2q = p + r]

$= (AR^{p-1})(AR^{r-1}) = a_p.a_r$ Hence, a_p, a_q, a_r are in G.P.

GEOMETRIC MEAN

Insertion of n Geometric Means Between a and b: Let a, b be two numbers, and let $G_1, G_2, \ldots\ldots G_n$ be n numbers such that a, $G_1, G_2, \ldots\ldots G_n$, b form a GP, then these n numbers are called n-Geometric means between a and b.

Then $b = (n + 2)^{th}$ term $= ar^{n+1}$. $\therefore$ $r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$ $\therefore$ $G_1 = ar, G_2 = ar^2 \ldots\ldots G_n = ar^n$

On putting the value of r, we can find the n geometric means between a and b.

Some important properties of Arithmetic and Geometric means between two given quantities:

Prop 1: If A and G are respectively arithmetic and geometric means between two positive numbers a and b, then A > G.

Proof: We have $A = \frac{a+b}{2}$ and G $\sqrt{ab}$

$\therefore A - G = \frac{a+b}{2} - \sqrt{ab} = \frac{a+b-2\sqrt{ab}}{2} = \frac{1}{2}[\sqrt{a} - \sqrt{b}^2] > 0$ $A = \frac{a+b}{2}$ A > G.

Prop II: If A and G are respectively arithmetic and geometric means between two positive quantities a and b, then the quadratic equation having a, b as its roots is $x^2 - 2Ax + G^2 = 0$

Proof: We have $A = \frac{a+b}{2}$ and $G = \sqrt{ab}$

The equation having a, b as its roots is

$x^2 - x(a + b) + ab = 0$

Or $x^2 - 2Ax + G^2 = 0$ $\left[\because A = \frac{a+b}{2} \text{ and } G = \sqrt{ab}\right]$

Prop III: If A and G be the A.M. and G.M. between two positive numbers, then the numbers are $A \pm \sqrt{A^2 - G^2}$

Proof: The equation having its roots as the given numbers is

$x^2 - 2Ax + G^2 = 0$

$\Rightarrow x = \frac{2A \pm \sqrt{4A^2 - 4G^2}}{2} \Rightarrow x = A \pm \sqrt{A^2 - G^2}$

Prop IV: Product of n G.M's between a and b = G^n, where G is the G.M between a and b i.e. G = $\sqrt{ab}$.

HARMONIC PROGRESSION (H. P.)

A series of quantities is said to be in harmonic progression when their reciprocals are in A. P. e .g. 1/3, 1/5, 1/7,…… are in H.P.

In general l/a, 1/(a. + d), 1/(a + 2d) ……….are in H.P.

The n^{th} term of an HP is the reciprocal of the n^{th} term of corresponding A.P.

HARMONIC MEAN

Insertion of n Harmonic Means between a and b: If a, b are two given numbers and between these numbers, n numbers $H_1, H_2, \ldots H_n$ are inserted such that a, H_1, H_2, b is a HP, then $H_1, H_2, \ldots H_n$ are called n Harmonic means between a and b.

As a, $H_1, H_2, \ldots H_n$, b, are in HP

$\therefore \frac{1}{a}, \frac{1}{H_1 H_2}, \ldots\ldots \frac{1}{H_n}, \frac{1}{b}$ are in AP $\therefore$ 1/b = 1/a + (n + 1) D therefore $D = \frac{a-b}{(n+1)ab}$

$\therefore \frac{1}{H_1} = \frac{1}{a} + D, \frac{1}{H_2}, \frac{1}{a} + 2D, \ldots\ldots, \frac{1}{H_n} = \frac{1}{a} + nD.$

On putting the value of D we can find the values of $H_1, H_2, \ldots H_n$.

Important results:

a. The HM of two numbers a and b is $H = \frac{2ab}{a+b}$

b. The Arithmetic mean A, Geometric mean G and Harmonic mean H of two numbers $AH = G^2$ i.e. A, G, H are in GP. Also $A \geq G \geq H$.

ARITHMETIC – GEOMETRIC SERIES

A series of the form a + (a + d) r + (a +2d) r^2 + (a + 3d) r^3 +…. is called an Arithmetic geometric series. The sum of n terms of an Arithmetic geometric series is $S_n = \frac{a}{1-r} + dr\frac{(1-r^{n-1})}{(1-r)^2} - \frac{[a+(n-1)d]r^n}{(1-r)}$.

If $|r| < 1$, then $r^n \to 0$ as $n \to \infty$

Arithmetic geometric series can be solved by the following method;

Ex. Find the sum of $1 + 2x + 3x^2 + 4x^2 + \ldots \infty$

Sol: The given series in an arithmetic-geometric series whose corresponding A.P. and G.P. are 1, 2, 3, 4,… and 1, x, x^2, x^3, … respectively. The common ratio of the G.P. is x. Let S_∞ denote the sum of the given series.

Then, $S_\infty = 1 + 2x + 3x^2 + 4x^2 + \ldots \infty$

$\Rightarrow x S_\infty = \quad x + 2x^2 + 3x^3 + \ldots \infty$

Subtracting (ii) from (i), we get

$S_\infty - x S_\infty = 1 + [x + x^2 + x^3 + \ldots \infty]$

$\Rightarrow S_\infty (1 - x) = 1 + \frac{x}{1-x} \Rightarrow S_\infty = \frac{1}{1-x} + \frac{x}{(1-x)^2} = \frac{1}{(1-x)^2}$

DIFFERENCE SERIES

If the differences between the successive terms are in A.P then it is called as difference series.

e.g.: 1, 3, 7, 13, 21, ……………

For such sequences, nth term = $An^2 + Bn + C$, where A, B, C are constants.